THE HILBERT SERIES OF THE RING ASSOCIATED TO AN ALMOST ALTERNATING MATRIX

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ABSTRACT. We give an explicit formula for the Hilbert Series of an algebra defined by a linearly presented, standard graded, residual intersection of a grade three Gorenstein ideal.

1. INTRODUCTION.

Finding explicit formulas for Hilbert series of residual intersections is a matter of serious concern; see, for example, [2, 3]. The first paper shows that there should be formulas which express the Hilbert series of a scheme in terms of the Hilbert series of its conormal modules. The second paper exhibits the explicit formulas and has applications to the dimension of secant varieties. The Hilbert series of the present paper are completely explicit; there is no need to use the Hilbert series of conormal modules.

The notion of residual intersection (see 1.3) was introduced by Artin and Nagata [1]; it has been improved and generalized by Huneke [7] and Huneke and Ulrich [8]. Residual intersections can be used to compute j-multiplicity [13], the dimension of secant varieties [3], complete intersection defect ideals [4], Segre classes of subschemes of projective space [6], and Chern numbers of smooth varieties [5].

The residual intersections that we consider all arise from an almost alternating matrix. If n and t are integers with n positive and t non-negative, then an $n\times(n+t)$ matrix ρ is called *almost alternating* if the left-most n columns of ρ form an alternating matrix. We are interested in non-square almost alternating matrices. Let R be a commutative Noetherian ring, n and t be positive integers, and ρ be an n by $(n+t)$ almost alternating matrix with entries in R. The matrix ρ gives rise to an ideal $J(\rho)$ in R, see (1.4). It is shown in [12] that $\operatorname{grade} J(\rho)\le t$ and if $\operatorname{grade} J(\rho)=t$, then $J(\rho)$ is a perfect ideal in R. It is also shown in [12] that, once minor hypotheses are imposed (see the statement of Corollary 1.2 for the details), then every residual intersection of a grade three Gorenstein ideal is equal to $J(\rho)$ for some almost alternating matrix ρ. Furthermore, a resolution $\mathcal{D}^0(\rho)$ of $R/J(\rho)$ by free

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R-modules is given in [12]; this resolution is minimal whenever the data permits such a claim.

Assume that R is a standard graded ring and that the entries of ρ are linear forms from R. In this paper we give explicit formulas for the Hilbert series and multiplicity of $\bar{R} = R/J(\rho)$. Recall that the Hilbert series of the graded ring $S = \bigoplus_{0\le i} S_i$ is the formal power series

$$\mathrm{HS}_S(z) = \textstyle\sum_i \lambda_{S_0}(S_i) z^i,$$

where $\lambda_{S_0}(\_)$ represents the length of an S_0-module, and the multiplicity of S is

$$e_S = (\dim S)! \lim_{i\to\infty} \frac{\lambda_S(S/\mathfrak{m}^i S)}{i^{\dim S}},$$

where $\mathfrak{m}$ is the maximal homogeneous ideal of S and "dim" represents Krull dimension.

Theorem 1.1 is the main result of the paper. (The ideal $J(\rho)$ is defined in (1.4).)

Theorem 1.1. *Let R_0 be an Artinian local ring, $R = \bigoplus_{0\le i} R_i$ be a standard graded R_0 algebra, n and t be positive integers, ρ be an $n \times (n+t)$ almost alternating matrix with homogeneous linear entries, J be the ideal $J(\rho)$, and $\bar{R}$ be the quotient ring R/J. If $t \le \operatorname{grade} J$, then the following statements hold.*

(a) *The Hilbert series of $\bar{R}$ is $\mathrm{HS}_{\bar{R}}(z) = \mathrm{HS}_R(z)\cdot(1-z)^t\cdot \mathrm{hn}_{\bar{R}}(z)$, where*

$$\mathrm{hn}_{\bar{R}}(z) = \sum_{i=0}^{n-1}\binom{t+i-1}{i}z^i - \sum_{i=\lfloor\frac{n+1}{2}\rfloor}^{n-1}\binom{t+2i-n-1}{2i-n}z^i. \tag{1.1.1}$$

(b) *The multiplicity, $e_{\bar{R}}$, of $\bar{R}$ is equal to*

$$e_R \cdot \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-2-2i+t}{t-1},$$

which is also equal to e_R times the number of monomials m of degree at most $n-1$ in t variables with $\deg m + n$ odd.

(c) *In particular, if R is a standard graded polynomial ring over a field, then the h-vector of $\bar{R}$ is the vector of coefficients of the polynomial $\mathrm{hn}_{\bar{R}}(z)$ and the multiplicity of $\bar{R}$ is*

$$\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-2-2i+t}{t-1}.$$

We highlight the application of Theorem 1.1 to residual intersections; see 1.3 for a definition of residual intersection.

Corollary 1.2. *Let R_0 be an Artinian local ring, $R = \bigoplus_{0\le i} R_i$ be a Cohen-Macaulay standard graded R_0-algebra, I be a linearly presented grade three Gorenstein ideal in R (presented by an alternating matrix), A be a sub-ideal of I minimally generated generated by t homogeneous elements for some t with $3 \le t$, J be the ideal $A :_R I$, and $\bar{R} = R/J$. Assume that*

(1) *the homogeneous minimal generators of A live in two degrees:* $\operatorname{init}(I)$ *and* $\operatorname{init}(I)+1$, *where* $\operatorname{init}(I)$ *is the least degree r for which I contains a non-zero element of degree r,*
(2) *the minimal number of generators of I/A is n for some positive integer n, and*
(3) *the ideal J is a t-residual intersection of I* (*that is,* $t \le \operatorname{grade} J$).

If either

(i) *the ring R is Gorenstein, or else,*
(ii) *the residual intersection $J = A :_R I$ is geometric,*

then conclusions (a), (b), *and* (c) *of the Theorem* 1.1 *give the Hilbert series and multiplicity of the ring $\bar{R}$.*

Remark. The multiplicity calculation of Corollary 1.2 is carried out for at least some residual intersections in [10]. Both calculations begin with the resolutions of [12] and both calculations involve some manipulation of binomial coefficients. The present calculation gives the entire h-vector of $\bar{R}$ in addition to the multiplicity (which is the sum of the entries in the h-vector).

1.3. Let R be a commutative Noetherian ring, I an ideal in R, t an integer with $\operatorname{ht} I \le t$, A a proper subideal of I which can be generated by t elements, and J the ideal $A :_R I$. If $t \le \operatorname{ht} J$, then J is called an *t-residual intersection* of I. If, furthermore, $I_P = A_P$ for all prime ideals P of R with $I \subseteq P$ and $\operatorname{ht} P \le t$, then J is called a *geometric t-residual intersection* of I.

1.4. Let $\rho = [X\ Y]$ be an almost alternating matrix, with X a square matrix. The alternating matrix which corresponds to ρ is

$$T = \begin{bmatrix} X & Y \\ -Y^{\mathrm{t}} & 0 \end{bmatrix}.$$

Let $J(\rho)$ be the ideal which is generated by the Pfaffians of all principal submatrices of T which contain X.

2. The proof of Theorem 1.1 and Corollary 1.2.

We begin by giving a bi-graded version of the complexes $\mathcal{D}^0(\rho)$ from [12, Sect.2]. The following result is not stated in [12]; but it could be. We actually make no use of the bi-homogeneous version of $\mathcal{D}^0(\rho)$; on the other hand, no extra work is involved in calculating the bi-graded twists rather than only the graded twists.

Lemma 2.1. *Let B be a bi-graded Noetherian ring, and n and t be positive integers, and $\rho = \begin{bmatrix} X & Y \end{bmatrix}$ be an almost alternating $n \times (n+t)$ with X an alternating matrix. Assume that each entry of X is bi-homogeneous of degree $(-1,0)$ and each entry of Y is bi-homogeneous of degree $(0,-1)$. Then the complex $\mathcal{D}^0(\rho)$ is*

$$0 \to \mathcal{D}_t \to \mathcal{D}_{t-1} \to \cdots \to \mathcal{D}_1 \to \mathcal{D}_0,$$

with

$$\mathcal{D}_N = \begin{cases} B, & \textit{if } N=0,\\ \bigoplus_{i=0}^{\lfloor\frac{n}{2}\rfloor} B(-i,2i-n)^{\binom{t}{n-2i}}, & \textit{if } N=1,\\ \bigoplus_{i=0}^{n-1} B(-i,i+1-N-n)^{\binom{N+n-2}{n-i-1}\binom{i+N-2}{i}\binom{t}{N+n-1-i}}, & \textit{if } 2\le N\le t, \textit{ and}\\ 0, & \textit{otherwise}. \end{cases}$$

Corollary 2.2. *If the hypotheses of the Theorem* 1.1 *are in effect, then*

$$\mathrm{HS}_{\bar{R}}(z)=\mathrm{HS}_R(z)\,\mathrm{HN}_{\bar{R}}(z), \tag{2.2.1}$$

for

$$\mathrm{HN}_{\bar{R}}(z)=1-\sum_{I=\lfloor\frac{n+1}{2}\rfloor}^{n}\binom{t}{2I-n}z^I+\sum_{I=n+1}^{n+t-1}(-1)^{I-n+1}\sum_{i=0}^{n-1}\binom{I-1}{n-i-1}\binom{i+I-n-1}{i}\binom{t}{I-i}z^I. \tag{2.2.2}$$

Proof. The hypotheses of Theorem 1.1, together with [12, Thm. 8.3], guarantee that the complex $\mathcal{D}^0(\rho)$ is a resolution of $\bar{R}$. One now reads from Lemma 2.1 that (2.2.1) holds with

$$\mathrm{HN}_{\bar{R}}(z)=1-\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\binom{t}{n-2i}z^{n-i}+\sum_{N=2}^{t}(-1)^N\sum_{i=0}^{n-1}\binom{N+n-2}{n-i-1}\binom{i+N-2}{i}\binom{t}{N+n-1-i}z^{N+n-1}.$$

Let $I=n-i$ in the first sum and $I=N+n-1$ in the second sum to obtain the stated formulation. $\square$

Proof of Theorem 1.1. We prove (a) by showing that $\mathrm{HN}_{\bar{R}}(z)=(1-z)^t\,\mathrm{hn}_{\bar{R}}(z)$, where $\mathrm{HN}_{\bar{R}}(z)$ and $\mathrm{hn}_{\bar{R}}(z)$ are the polynomials given in (2.2.2) and (1.1.1), respectively. An easy calculation yields that $(1-z)^t\,\mathrm{hn}_{\bar{R}}(z)$ is equal to

$$-\sum_{I=\lfloor\frac{n+1}{2}\rfloor}^{n+t-1}\left(\sum_{i=\lfloor\frac{n+1}{2}\rfloor}^{n-1}(-1)^{I-i}\binom{t+2i-n-1}{2i-n}\binom{t}{I-i}\right)z^I+\sum_{I=0}^{n+t-1}\left(\sum_{i=0}^{n-1}(-1)^{I-i}\binom{t+i-1}{i}\binom{t}{I-i}\right)z^I;$$

consequently, it suffices to prove

$$1-\sum_{I=\lfloor\frac{n+1}{2}\rfloor}^{n}\binom{t}{2I-n}z^I=\begin{cases}-\sum_{I=\lfloor\frac{n+1}{2}\rfloor}^{n}\left(\sum_{i=\lfloor\frac{n+1}{2}\rfloor}^{n-1}(-1)^{I-i}\binom{t+2i-n-1}{2i-n}\binom{t}{I-i}\right)z^I\\ +\sum_{I=0}^{n}\left(\sum_{i=0}^{n-1}(-1)^{I-i}\binom{t+i-1}{i}\binom{t}{I-i}\right)z^I\end{cases} \tag{2.2.3}$$

and

$$\begin{aligned}&\sum_{I=n+1}^{n+t-1}(-1)^{I-n+1}\sum_{i=0}^{n-1}\binom{I-1}{n-i-1}\binom{i+I-n-1}{i}\binom{t}{I-i}z^I\\ &=\sum_{I=n+1}^{n+t-1}\left(-\sum_{i=\lfloor\frac{n+1}{2}\rfloor}^{n-1}(-1)^{I-i}\binom{t+2i-n-1}{2i-n}\binom{t}{I-i}+\sum_{i=0}^{n-1}(-1)^{I-i}\binom{t+i-1}{i}\binom{t}{I-i}\right)z^I.\end{aligned} \tag{2.2.4}$$

The equation (2.2.3) is established in Lemma 2.3. Remove the variable z from (2.2.4), multiply by $(-1)^{I+n+1}$, and remove the unnecessary constraints on the index i. (Indeed, $\binom{a}{b}$ is zero if b is negative.) To establish (2.2.4), it suffices to show that if $1 \le I$, then

$$0 = \begin{cases} -\sum\limits_{i\in\mathbb{Z}} \binom{I-1}{n-i-1}\binom{i+I-n-1}{i}\binom{t}{I-i} \\ +\sum\limits_{i\le n-1} (-1)^{n-i}\binom{t+2i-n-1}{2i-n}\binom{t}{I-i} \\ +\sum\limits_{i\le n-1} (-1)^{n+1-i}\binom{t+i-1}{i}\binom{t}{I-i}. \end{cases} \tag{2.2.5}$$

The right side of (2.2.5) is called $Q(n-1,t,I,0)$ in Definition 3.1. It is shown in Proposition 3.2 that $0 = Q(n-1,t,I,0)$. The hypotheses ($0 \le t$, $0 \le n-1$, and $1 \le I$) of Proposition 3.2 are satisfied by the present data.

To prove (b), it suffices to calculate

$$\begin{aligned} \mathrm{hn}_{\bar{R}}(1) &= \sum_{i=0}^{n-1}\binom{t+i-1}{i} - \sum_{i=\lfloor\frac{n+1}{2}\rfloor}^{n-1}\binom{t+2i-n-1}{2i-n} \\ &= \binom{t+n-1}{n-1} - \sum_{i\le n-1}\binom{t+2i-n-1}{2i-n}. \end{aligned}$$

Let $j = 2i-n$ to obtain

$$\mathrm{hn}_{\bar{R}}(1) = \binom{t+n-1}{n-1} - \sum_{\substack{j\le n-2 \\ j+n \text{ even}}}\binom{t+j-1}{j}. \tag{2.2.6}$$

The binomial coefficient $\binom{t+n-1}{n-1}$ is equal to the number of monomials of degree at most $n-1$ in t variables and the sum on the right side of (2.2.6) is the number of monomials m of degree at most $n-2$ in t variables with $\deg m + n$ even. The difference is the number of monomials m of degree at most $n-1$ in t variables with $\deg m + n$ odd.

Assertion (c) requires no further proof. □

Lemma 2.3. *If n and t are positive integers, then* (2.2.3) *holds.*

Proof. The binomial coefficient $\binom{t}{I-i}$ is zero unless $i \le I$; consequently, if the upper limit for i on the right side of (2.2.3) is changed from $n-1$ to n, the only value of I which is affected is $I = n$, and, if $I = i = n$, then $\binom{t+i-1}{i} = \binom{t+2i-n-1}{2i-n}$. It suffices to show

$$1 - \sum_{I=\lfloor\frac{n+1}{2}\rfloor}^{n}\binom{t}{2I-n}z^I = \begin{cases} \sum\limits_{I=0}^{n}\Big(\sum\limits_{i=0}^{n}(-1)^{I-i}\binom{t}{I-i}\binom{t+i-1}{i}\Big)z^I \\ -\sum\limits_{I=\lfloor\frac{n+1}{2}\rfloor}^{n}\Big(\sum\limits_{i=\lfloor\frac{n+1}{2}\rfloor}^{n}(-1)^{I-i}\binom{t}{I-i}\binom{t+2i-n-1}{2i-n}\Big)z^I. \end{cases} \tag{2.3.1}$$

Observe next that

$$1 = \sum_{I=0}^{n}\Big(\sum_{i=0}^{n}(-1)^{I-i}\binom{t}{I-i}\binom{t+i-1}{i}\Big)z^I. \tag{2.3.2}$$

Indeed, the right side of (2.3.2) is equal to the first $n+1$ terms of the power series expansion of

$$1=(1-z)^t\frac{1}{(1-z)^t}.$$

Subtract (2.3.2) from (2.3.1); multiply by -1; and look at one coefficient at a time. Fix I with $\lfloor\frac{n+1}{2}\rfloor\le I\le n$. It suffices to show that

$$\binom{t}{2I-n}=\sum_{i=\lfloor\frac{n+1}{2}\rfloor}^{n}(-1)^{I-i}\binom{t}{I-i}\binom{t+2i-n-1}{2i-n}. \tag{2.3.3}$$

The right side of (2.3.3) is

$$\begin{aligned}
&=\sum_{i=\lfloor\frac{n+1}{2}\rfloor}^{I}(-1)^{I-i}\binom{t}{I-i}\binom{t+2i-n-1}{2i-n}, &&\text{because } \binom{t}{I-i}=0 \text{ if } I<i\text{, and } I\le n,\\
&=\sum_{j=0}^{\lfloor\frac{2I-n}{2}\rfloor}(-1)^{j}\binom{t}{j}\binom{2I-n-2j+t-1}{2I-n-2j}, &&\text{with } j=I-i.
\end{aligned}$$

Thus, the right side of (2.3.3) is

$$\begin{aligned}
&=\sum_{j=0}^{\lfloor\frac{2I-n}{2}\rfloor}(\text{the coefficient of } z^{2j} \text{ in } (1-z^2)^t)\cdot(\text{the coefficient of } z^{2I-n-2j} \text{ in } \tfrac{1}{(1-z)^t})\\
&=\text{the coefficient of } z^{2I-n} \text{ in } \tfrac{(1-z^2)^t}{(1-z)^t}=(1+z)^t\\
&=\binom{t}{2I-n},
\end{aligned}$$

which is the left side of (2.3.3). □

Proof of Corollary 1.2. The Corollary is an immediate consequence of Theorem 1.1 and [12, 10.2]. We offer the following explanation of how the relevant $n\times(n+t)$ almost alternating matrix arises. Let μ be the minimal number of generators of I. One starts with a $\mu\times\mu$ matrix X of linear forms which presents I and a $\mu\times t$ matrix Y which expresses the generators of A in terms of the generators of I. One can arrange this data so that the matrix Y has the form

$$Y=\begin{bmatrix}Y' & 0\\ 0 & I_{\mu-n}\end{bmatrix},$$

where the entries of Y' are linear forms and $I_{\mu-n}$ is the identity matrix with $\mu-n$ rows and columns. The ideal J is $J(\rho)$, where ρ is the $\mu\times(\mu+t)$ almost alternating matrix $\begin{bmatrix}X & Y\end{bmatrix}$. The ideal J is also $J(\rho')$, where ρ' is the $n\times(n+t)$ almost alternating matrix of linear forms which is obtained from ρ by deleting the last $\mu-n$ rows and columns of ρ. □

3. A FAMILY OF IDENTITIES

The calculations in this section are inspired by the Hilbert series calculations in [11].

The identities

$$Q(w,t,I,0)=0 \tag{3.0.1}$$

of Proposition 3.2 are crucial to the proof of Theorem 1.1. The integers $Q(w,t,I,\alpha)$, with $1\le\alpha$, are introduced in order to prove (3.0.1). We created $Q(*,t,*,\alpha+1)$ to be the first difference function $Q(*,t+1,*,\alpha)-Q(*,t,*,\alpha)$ of the the function $Q(*,t,*,\alpha)$. Fortunately, we are able to find a closed formula for $Q(*,t,*,\alpha+1)$ and thereby verify that this first difference function satisfies the desired initial condition $Q(*,0,*,\alpha+1)=0$.

Definition 3.1. For integers w,t,I,α, with $0\le\alpha$, define $Q(w,t,I,\alpha)$ to be the integer

$$Q(w,t,I,0)=\begin{cases}-\sum\limits_{i\in\mathbb{Z}}\binom{I-1}{w-i}\binom{i+I-w-2}{i}\binom{t}{I-i}\\ +\sum\limits_{i\le w}(-1)^{i+w+1}\binom{2i-w-2+t}{2i-w-1}\binom{t}{I-i} & \text{if } \alpha=0, \text{ and}\\ +\sum\limits_{i\le w}(-1)^{i+w}\binom{t+i-1}{i}\binom{t}{I-i},\end{cases}$$

$$Q(w,t,I,\alpha)=\begin{cases}-\sum\limits_{i\in\mathbb{Z}}\binom{I-1}{w-i}\binom{i+I-w-2}{i}\binom{t}{I-i-\alpha}\\ +\sum\limits_{i\le w}(-1)^{i+w+1+\alpha}\binom{2i-w-2+t+\alpha}{2i-w-1+\alpha}\binom{t}{I-i-\alpha} & \text{if } 1\le\alpha.\\ +\sum\limits_{i=1}^{\alpha-1}(-1)^{i+1}\binom{I-1}{w+i}\binom{t+w+i}{I-\alpha+i},\end{cases}$$

Proposition 3.2. *If w, t, I, and α are integers with α, t, $w+\alpha$, and $I-1$ all non-negative, then $Q(w,t,I,\alpha)=0$.*

Proof. Fix integers w, I, and α with α, $w+\alpha$, and $I-1$ non-negative. We show in Lemmas 3.3 and 3.4 that

$$Q(w,t+1,I,\alpha)=Q(w,t,I,\alpha)+Q(w,t,I,\alpha+1) \tag{3.2.1}$$

$$Q(w,0,I,\alpha)=0. \tag{3.2.2}$$

This recurrence relation now yields that $Q(w,t,I,\alpha)=0$ for all non-negative t. $\square$

Lemma 3.3. *If w, t, I, and α are integers with t, α, $w+\alpha$, and $I-1$ non-negative, then equality holds in* (3.2.1).

Proof. We treat the cases $\alpha=0$ and $1\le\alpha$ separately. We begin with $\alpha=0$ and we compute

$$Q(w,t,I,1)-Q(w,t+1,I,0)+Q(w,t,I,0). \tag{3.3.1}$$

Write

$$Q(w,t,I,1)=S_1+S_2,\ -Q(w,t+1,I,0)=S_3+S_4+S_5, \text{ and } Q(w,t,I,0)=S_6+S_7+S_8,$$

for

$$S_1=-\sum_{i\in\mathbb{Z}}\binom{I-1}{w-i}\binom{i+I-w-2}{i}\binom{t}{I-i-1},$$

$$\begin{aligned}
S_2 &= \sum_{i\le w}(-1)^{i+w}\tbinom{2i-w-1+t}{2i-w}\tbinom{t}{I-i-1},\\
S_3 &= \sum_{i\in\mathbb{Z}}\tbinom{I-1}{w-i}\tbinom{i+I-w-2}{i}\tbinom{t+1}{I-i},\\
S_4 &= -\sum_{i\le w}(-1)^{i+w+1}\tbinom{2i-w-1+t}{2i-w-1}\tbinom{t+1}{I-i},\\
S_5 &= -\sum_{i\le w}(-1)^{i+w}\tbinom{t+i}{i}\tbinom{t+1}{I-i},\\
S_6 &= -\sum_{i\in\mathbb{Z}}\tbinom{I-1}{w-i}\tbinom{i+I-w-2}{i}\tbinom{t}{I-i},\\
S_7 &= \sum_{i\le w}(-1)^{i+w+1}\tbinom{2i-w-2+t}{2i-w-1}\tbinom{t}{I-i}, \text{ and}\\
S_8 &= \sum_{i\le w}(-1)^{i+w}\tbinom{t+i-1}{i}\tbinom{t}{I-i}.
\end{aligned}$$

Apply the Pascal identity

$$\tbinom{a}{b} = \tbinom{a-1}{b} + \tbinom{a-1}{b-1}, \tag{3.3.2}$$

which holds for all integers a and b, to see that $S_1+S_3+S_6=0$. Apply the Pascal identity three times: first to write $S_4=S_4'+S_4''$ with

$$\begin{aligned}
S_4' &= -\sum_{i\le w}(-1)^{i+w+1}\tbinom{2i-w-1+t}{2i-w-1}\tbinom{t}{I-i-1} \text{ and}\\
S_4'' &= -\sum_{i\le w}(-1)^{i+w+1}\tbinom{2i-w-1+t}{2i-w-1}\tbinom{t}{I-i},
\end{aligned}$$

and then to obtain

$$\begin{aligned}
S_2+S_4' &= \sum_{i\le w}(-1)^{i+w}\tbinom{2i-w+t}{2i-w}\tbinom{t}{I-i-1} \text{ and}\\
S_4''+S_7 &= \sum_{i\le w}(-1)^{i+w}\tbinom{2i-w-2+t}{2i-w-2}\tbinom{t}{I-i}.
\end{aligned}$$

Replace the index i in $S_4''+S_7$ by $i+1$ and then combine with S_2+S_4'. The result is

$$S_2+S_4+S_7 = \tbinom{t+w}{w}\tbinom{t}{I-w-1}.$$

Write $S_5=S_5'+S_5''$ with

$$\begin{aligned}
S_5' &= -\sum_{i\le w}(-1)^{i+w}\tbinom{t+i}{i}\tbinom{t}{I-i-1},\\
S_5'' &= -\sum_{i\le w}(-1)^{i+w}\tbinom{t+i}{i}\tbinom{t}{I-i},
\end{aligned}$$

Observe that

$$S_5''+S_8 = -\sum_{i\le w}(-1)^{i+w}\tbinom{t+i-1}{i-1}\tbinom{t}{I-i}.$$

Replace the index i in $S_5''+S_8$ by $i+1$ and then combine with S_5'. The result is

$$S_5+S_8 = -\tbinom{t+w}{w}\tbinom{t}{I-w-1}.$$

Thus, (3.3.1), which is equal to $\sum_{i=1}^{8} S_i$, is also equal to $S_1+S_3+S_6$ plus $S_2+S_4+S_7$ plus S_5+S_8, and this is zero. The assertion holds when $\alpha=0$. In this part of the argument, we did not need to use the conditions imposed on t, w, and I.

The $1\le\alpha$ part of the argument is similar, but with important differences. We compute

$$Q(w,t,I,\alpha+1)-Q(w,t+1,I,\alpha)+Q(w,t,I,\alpha). \tag{3.3.3}$$

Write $Q(w,t,I,\alpha+1)=\sum_{i=1}^{3} S_1$, $-Q(w,t+1,I,\alpha)=\sum_{i=4}^{6} S_i$ and $Q(w,t,I,\alpha)=\sum_{i=7}^{9} S_i$, as before. Observe that $S_1+S_4+S_7=0$,

$$\begin{aligned} S_2+S_5+S_8 &= (-1)^{\alpha}\tbinom{w+t+\alpha}{w+\alpha}\tbinom{t}{I-w-\alpha-1}, \text{ and}\\ S_3+S_6+S_9 &= (-1)^{\alpha+1}\tbinom{I-1}{w+\alpha}\tbinom{t+w+\alpha}{I-1}. \end{aligned}$$

Thus, (3.3.3) is equal to

$$(-1)^{\alpha}\left[\tbinom{w+t+\alpha}{w+\alpha}\tbinom{t}{I-w-\alpha-1}-\tbinom{I-1}{w+\alpha}\tbinom{t+w+\alpha}{I-1}\right]. \tag{3.3.4}$$

The hypotheses guarantee that $w+t+\alpha$, $w+\alpha$, t, and $I-1$ all are non-negative. Furthermore, we observe that if $I-w-\alpha-1<0$ or $t<I-w-\alpha-1$, then both summands in (3.3.4) are zero. Henceforth, we may assume $0\le I-w-\alpha-1$ or $I-w-\alpha-1\le t$. In this case, each binomial coefficient $\binom{a}{b}$ in (3.3.4) satisfies $0\le b\le a$; consequently, $\binom{a}{b}=\frac{a!}{b!(a-b)!}$ and a straightforward calculation shows that (3.3.4) is zero. $\square$

Lemma 3.4. *If w, I, and α are integers with α, $w+\alpha$, and $I-1$ non-negative, then equality holds in* (3.2.2).

Proof. We evaluate $Q(w,0,I,\alpha)=S_1+S_2+S_3$, with

$$\begin{aligned} S_1 &= -\sum_{i\in\mathbb{Z}}\tbinom{I-1}{w-i}\tbinom{i+I-w-2}{i}\tbinom{0}{I-i-\alpha},\\ S_2 &= \sum_{i\le w}(-1)^{i+w+1+\alpha}\tbinom{2i-w-2+\alpha}{2i-w-1+\alpha}\tbinom{0}{I-i-\alpha}, \text{ and}\\ S_3 &= \sum_{i=1}^{\alpha-1}(-1)^{i+1}\tbinom{I-1}{w+i}\tbinom{w+i}{I-\alpha+i}. \end{aligned}$$

The binomial coefficient $\binom{0}{b}$ is zero unless $b=0$. If "S" is a statement, then we write $\chi(S)$ to mean

$$\chi(S)=\begin{cases}1 & \text{if "}S\text{" is true}\\ 0 & \text{otherwise.}\end{cases}$$

We see that

$$\begin{aligned} S_1 &= -\tbinom{I-1}{w-I+\alpha}\tbinom{2I-\alpha-w-2}{I-\alpha} \qquad \text{and}\\ S_2 &= \chi(I-\alpha\le w)(-1)^{I+w+1}\tbinom{2I-\alpha-w-2}{2I-\alpha-w-1}. \end{aligned} \tag{3.4.1}$$

The binomial coefficient $\binom{b-1}{b}$ is zero unless $b=0$. Furthermore, if $w=2I-\alpha-1$, then $I-\alpha\le w$ holds automatically because $0\le I-1$. It follows that

$$S_2=(-1)^{I+w+1}\chi(w=2I-\alpha-1). \tag{3.4.2}$$

Let $k=w+i$ in S_3 in order to obtain

$$S_3=\sum_{k=w+1}^{w+\alpha-1}(-1)^{k-w+1}\binom{I-1}{k}\binom{k}{I-\alpha-w+k}=S_3'+S_3''+S_3''',$$

with

$$\begin{aligned} S_3' &= \sum_{k\in\mathbb{Z}}(-1)^{k-w+1}\binom{I-1}{k}\binom{k}{I-\alpha-w+k}\\ S_3'' &= \sum_{k\le w}(-1)^{k-w}\binom{I-1}{k}\binom{k}{I-\alpha-w+k}\\ S_3''' &= \sum_{w+\alpha\le k}(-1)^{k-w}\binom{I-1}{k}\binom{k}{I-\alpha-w+k}. \end{aligned}$$

For S_3', we use the identity

$$\sum_{k\in\mathbb{Z}}(-1)^k\binom{b+k}{c+k}\binom{a}{k}=(-1)^a\binom{b}{a+c},$$

which holds for integers a, b, and c with $0\le a$. (See, for example, [9, Lemma 1.3].) Take $a=I-1$, $b=0$, and $c=I-\alpha-w$. We conclude that

$$S_3'=(-1)^{w+I}\binom{0}{2I-1-\alpha-w}=(-1)^{w+I}\chi(w=2I-1-\alpha)=-S_2, \tag{3.4.3}$$

for S_2 given in (3.4.2). In S_3''', the ambient hypothesis guarantees that $0\le I-1$. If the term corresponding to k is non-zero, then

$$0\le k\le I-1<I\le I-\alpha-w+k\le k.$$

The inequality $k<k$ never occurs; consequently,

$$S_3'''=0. \tag{3.4.4}$$

Notice that S_3'' is zero unless $0\le I-\alpha$. So,

$$\begin{aligned} S_3'' &= \chi(0\le I-\alpha)\sum_{k\le w}(-1)^{k-w}\binom{I-1}{k}\binom{k}{I-\alpha-w+k}\\ &= \chi(0\le I-\alpha)\sum_{k=\max\{0,w+\alpha-I\}}^{\min\{w,I-1\}}(-1)^{k-w}\binom{I-1}{k}\binom{k}{I-\alpha-w+k}. \end{aligned}$$

We next show that

$$S_3''=\chi(0\le I-\alpha)\binom{I-1}{w+\alpha-I}\sum_{k=\max\{0,w+\alpha-I\}}^{\min\{w,I-1\}}(-1)^{k-w}\binom{2I-1-\alpha-w}{I-1-k}. \tag{3.4.5}$$

Observe that if $w+\alpha-I<0$, then both sides of (3.4.5) are zero; and therefore, in order to establish (3.4.5), it suffices to prove that

$$\binom{I-1}{k}\binom{k}{I-\alpha-w+k}=\binom{I-1}{w+\alpha-I}\binom{2I-1-\alpha-w}{I-1-k} \tag{3.4.6}$$

when

$$\max\{0,w+\alpha-I\}\le k\le\min\{w,I-1\}\quad\text{and}\quad 0\le w+\alpha-I. \tag{3.4.7}$$

The hypotheses (3.4.7) guarantee that each of the four binomial coefficients $\binom{a}{b}$ from (3.4.6) satisfies $0 \le a \le b$; consequently each of these binomial coefficients is equal to $\frac{a!}{b!(a-b)!}$. At this point (3.4.6) can be established with no difficulty; and therefore, the equation (3.4.5) has been established.

Let $\ell = I-1-k$ in (3.4.5) to see that

$$S_3'' = \chi(0 \le I-\alpha)\binom{I-1}{w+\alpha-I}\sum_{\ell=\max\{0,I-1-w\}}^{\min\{2I-1-w-\alpha,I-1\}}(-1)^{I-1-w-\ell}\binom{2I-1-\alpha-w}{\ell}$$

The constraint $0 \le \ell$ is not needed because $\binom{2I-1-\alpha-w}{\ell} = 0$ when $\ell < 0$. On the other hand, as we think about S_3'' we may as well assume $0 \le w+\alpha-I$ (otherwise $S_3''=0$). It follows that $I-w-\alpha \le 0$ and $2I-1-w-\alpha \le I-1$. Thus,

$$S_3'' = \chi(0 \le I-\alpha)\binom{I-1}{w+\alpha-I}\sum_{\ell=I-1-w}^{2I-1-w-\alpha}(-1)^{I-1-w-\ell}\binom{2I-1-\alpha-w}{\ell} = T_1+T_2,$$

for

$$T_1 = \chi(0 \le I-\alpha)\binom{I-1}{w+\alpha-I}(-1)^{I-1-w}\sum_{\ell \le 2I-1-w-\alpha}(-1)^{\ell}\binom{2I-1-\alpha-w}{\ell}$$

and

$$T_2 = -\chi(0 \le I-\alpha)\binom{I-1}{w+\alpha-I}(-1)^{I-1-w}\sum_{\ell \le I-2-w}(-1)^{\ell}\binom{2I-1-\alpha-w}{\ell}.$$

(The factor $\chi(0 \le I-\alpha)$ is very important. If $I-\alpha$ is less than zero, then S_3'' is zero, but

$$\binom{I-1}{w+\alpha-I}(-1)^{I-1-w}\left[\sum_{\ell \le 2I-1-w-\alpha}(-1)^{\ell}\binom{2I-1-\alpha-w}{\ell} - \sum_{\ell \le I-2-w}(-1)^{\ell}\binom{2I-1-\alpha-w}{\ell}\right]$$

is not necessarily zero.) Apply Lemma 3.5 to see that

$$\begin{aligned} T_1 &= \chi(0 \le I-\alpha)\binom{I-1}{w+\alpha-I}(-1)^{I-\alpha}\binom{2I-w-\alpha-2}{2I-w-\alpha-1} \text{ and} \\ T_2 &= \chi(0 \le I-\alpha)\binom{I-1}{w+\alpha-I}\binom{2I-2-\alpha-w}{I-2-w}. \end{aligned}$$

The binomial coefficient $\binom{b-1}{b}$ is zero unless $b=0$ and $\binom{I-1}{I-1}=1$, since $0 \le I-1$; therefore,

$$T_1 = (-1)^{I-\alpha}\chi(0 \le I-\alpha)\chi(w = 2I-1-\alpha). \tag{3.4.8}$$

The integer T_2 is zero unless $0 \le I-2-w$. If $0 \le I-2-w$, then

$$0 \le (I-\alpha)+(I-2-w) = 2I-\alpha-2-w$$

and $\binom{2I-2-\alpha-w}{I-2-w} = \binom{2I-2-\alpha-w}{I-\alpha}$. Thus,

$$T_2 = \chi(0 \le I-\alpha)\chi(0 \le I-2-w)\binom{I-1}{w+\alpha-I}\binom{2I-2-\alpha-w}{I-\alpha}.$$

The factor $\binom{2I-2-\alpha-w}{I-\alpha}$ of T_2 makes the factor $\chi(0\le I-\alpha)$ redundant; hence,

$$T_2 = \quad \chi(0\le I-2-w)\binom{I-1}{w+\alpha-I}\binom{2I-2-\alpha-w}{I-\alpha}.$$

Recall the integer S_1 from (3.4.1) and observe that

$$S_1+T_2 = -\chi(I-1\le w)\binom{I-1}{w+\alpha-I}\binom{2I-2-\alpha-w}{I-\alpha}.$$

Apply the identity $\binom{a}{b}=(-1)^b\binom{b-a-1}{b}$, which holds for all integers a and b, to write

$$S_1+T_2 = -(-1)^{I-\alpha}\chi(I-1\le w)\binom{I-1}{w+\alpha-I}\binom{w+1-I}{I-\alpha}.$$

If S_1+T_2 is non-zero, then $0\le I-1$ and $0\le w+1-I$. Thus, if S_1+T_2 is non-zero, then

$$w+\alpha-I\le I-1 \quad \text{and} \quad I-\alpha\le w+1-I;$$

$$w+\alpha-I\le I-1\le w+\alpha-I;$$

and $w=2I-\alpha-1$. We have shown that

$$\begin{aligned} S_1+T_2 &= -(-1)^{I-\alpha}\chi(w=2I-\alpha-1)\chi(I-1\le w)\binom{I-1}{w+\alpha-I}\binom{w+1-I}{I-\alpha} \\ &= -(-1)^{I-\alpha}\chi(w=2I-\alpha-1)\chi(\alpha\le I)\binom{I-1}{I-1}\binom{I-\alpha}{I-\alpha} \\ &= -(-1)^{I-\alpha}\chi(w=2I-\alpha-1)\chi(\alpha\le I). \end{aligned}$$

Recall the value of T_1 from (3.4.8). We have shown that

$$S_1+T_2=-T_1. \tag{3.4.9}$$

Combine (3.4.9), (3.4.3), and (3.4.4) to see that $Q(w,0,I,\alpha)$, which is equal to

$$(S_1+T_1+T_2)+(S_2+S_3')+S_3''',$$

is zero. □

Lemma 3.5. *If a and b are integers, then*

$$\sum_{\ell\le b}(-1)^\ell\binom{a}{\ell}=(-1)^b\binom{a-1}{b}.$$

Proof. If $b<0$, then both sides are zero. If $b=0$, then both sides are 1. A short induction completes the argument for $0<b$. The value of a is not relevant; Pascal's identity (3.3.2) holds for all integers. □

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